

Multi-directed animals, connected heaps of dimers and Lorentzian triangulations¹

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to my friend Tony Guttmann for his 60th birthday

Abstract. Bousquet-Mélou and Rechnitzer have introduced the class of multi-directed animals, as an extension of the classical 2D directed animals. They gave an explicit expression for their generating function. In the case of directed animals, the corresponding generating function is algebraic and various “combinatorial explanations” (bijective proofs) have been given, in particular using the so-called model of heaps of dimers. Although the generating function for multi-directed animals is not algebraic, even worst not D-finite, we give a bijective proof of Bousquet-Mélou and Rechnitzer’s formula, introducing the “Nordic decomposition” of a connected heap of dimers. One possible interest of this bijective proof is in relation with 2D Lorentzian quantum gravity. Ambjørn, Loll, Di Francesco, Guitter and Kristjansen have introduced and studied the notion of Lorentzian triangulations. There exist correspondences between these triangulations, connected heaps of dimers and multi-directed animals.

1. Introduction: directed animals and heaps of dimers

The enumeration of *animals* (or *polyominoes*) is a longstanding problem in combinatorics and in statistical physics. These objects have been intensively studied for more than 40 years, both by physicists and combinatorists. Some asymptotic results are known for the general case, see for example numerical studies by Jensen and Guttmann [16]. Exact enumerative results have been given for some subclasses of polyominoes defined essentially by some conditions of *directedness* or *convexity*. A *directed animal* on the square (resp. triangular) lattice is a set of points (or *cells*) such that any cell of the animal can be reached from a fixed cell, called the *source*, by a *path* that only visits cells of the animal, and such that each *elementary step* is going North or East (resp. North, East or North-East) to the next neighbour. Directed animals on the square and on the triangular lattice are shown on Figure 1.

Let a_n and b_n be the number of directed animals having n cells, drawn on respectively the square and triangular lattice. Let $P_S(t)$ and $P_T(t)$ be the corresponding generating functions, that is

$$P_S(t) = \sum_{n \geq 1} a_n t^n ; P_T(t) = \sum_{n \geq 1} b_n t^n. \quad (1.1)$$

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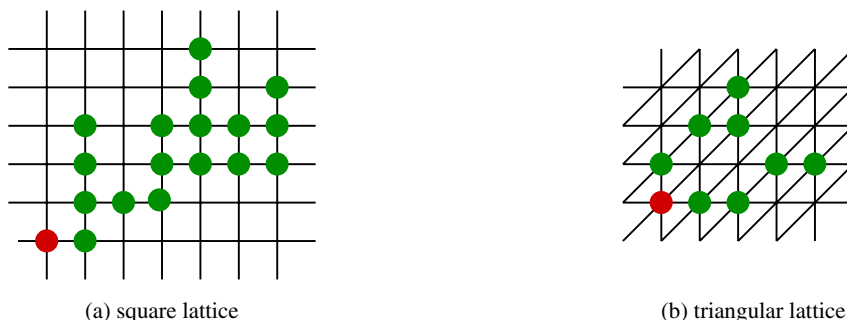


Figure 1. Directed animal.

Various solutions for the enumeration of directed animals have been given, Dhar [9], [10], Hakim, Nadal [13] (following Nadal, Derrida, Vannimenus [17]), Viennot [20], Gouyou-Beauchamps, Viennot [12], Barcucci and al [2], B  tr  ma, Penaud [3], [4], Penaud [18], Shapiro [19]. The generating functions $P_S(t)$ and $P_T(t)$ are algebraic generating functions and simple formulae can be given, related to Catalan and Motzkin numbers:

$$P_S(t) = \frac{1}{2} \left(\sqrt{\frac{1+t}{1-3t}} - 1 \right) ; P_T(t) = \frac{1}{2} \left(\frac{1}{\sqrt{1-4t}} - 1 \right). \tag{1.2}$$

Some solutions are based on the combinatorial theory of *heaps of pieces* introduced by the author in [21]. In this paper we will only use *heaps of dimers* on the discrete line of integers \mathbb{Z} . We shall recall here intuitively the definitions and the basic properties we need. Intuitively, a heap of dimers is obtained by dropping a finite number of dimers towards an horizontal axis. Each *dimer* connects two consecutive vertical lines, and fall until it touches the horizontal axis or another dimer. We can imagine that each dimer is a solid piece having thickness of one unit, so that dimers are lying at heights which are positive integers. See Figure 2.

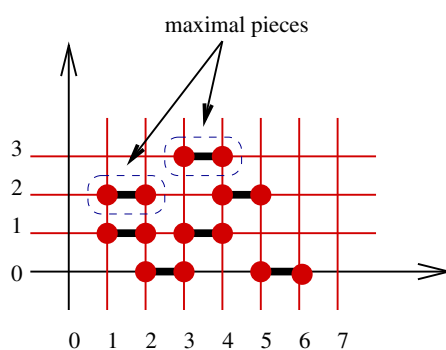


Figure 2. A heap of dimers.

For more precise or mathematical definitions the reader may consult Viennot [20], [21] where the general theory is introduced. Heaps of pieces form a kind of geometric and combinatorial model, interpreting the so-called *commutation monoids* introduced by Cartier and Foata in [8]. See also Bousquet-M  lou and Viennot [6] for another definition of heaps of pieces.

Heaps of dimers correspond to monoids generated by the variables $a_i, i \in \mathbb{Z}$, with the following commutation rules

$$a_i a_j = a_j a_i \text{ for } |i - j| \geq 2, i, j \in \mathbb{Z}. \tag{1.3}$$

A *maximal dimer* is a dimer that can be removed from the heap by sliding it upwards without bumping other dimers. A *pyramid* is a heap having only one maximal dimer. A *half-pyramid* is a pyramid such that the maximal dimer lies in the leftmost non empty column.

Push operator

This operator will appear several times in this paper. Take an arbitrary heap H and a particular piece α of H . Imagine we take this piece α and push it down. In the process, α may bump into other dimers, which may bump into other dimers, etc ... Intuitively, a whole pyramid $P(\alpha)$, with α as maximal piece will be extracted from the heap H . What will remain will be another heap $G(\alpha)$. We have decomposed the heap H into a heap G and a pyramid P . In fact, for the reader more familiar with heaps of pieces theory, we can write H as a *non-ambiguous product* $P \otimes G$, where the product means intuitively to “put the heap G above the heap P ”. This product is in fact isomorphic to the product in the associated commutation monoid, see [21]. The important fact to remember is that the *generating function* for the heap H will be the product of the generating function corresponding to G and P . Here, the generating function of heaps will be the generating function corresponding to the enumeration of heaps according to the number of dimers.

Now, as introduced in [20] or [21], there exist a simple bijection between directed animals on triangular lattice and pyramids of dimers (up to translation), as shown on Figure 3. We rotate the directed animal A so that the source point is upwards and the North-East direction becomes the South direction. We replace each cell of the animal by a dimer and we let them fall as shown on Figure 3. We get a pyramid $P = V(A)$. The map V is a bijection. In the case of square lattice animals, the map V gives a bijection between animals on a square lattice and *strict* pyramids of dimers. There are pyramids such that no two dimers are just one over another. Note that this local configuration in a heap corresponds to a North-East elementary step in the directed animal.

We resume the simple solution of [4] or [18] explaining the algebraic nature of P_S and P_T . It is simpler to explain it in the case of a triangular lattice. Let P be a pyramid. If P is not a half-pyramid, then we apply the push operator to the maximal dimer in the (non empty) column just at the left of the column containing the maximal piece of P , (see Figure 4). Similarly, any half-pyramid can be decomposed into product of one or two half-pyramids, as shown on Figure 4.

From these decompositions, we get the following algebraic system of equations for the generating functions $P(t)$ and $Q(t)$ of pyramids and half-pyramids on a triangular lattice.

$$\begin{cases} P(t) = Q(t) + P(t)Q(t), \\ Q(t) = t + 2tQ(t) + tQ^2(t). \end{cases} \tag{1.4}$$

In the case of square lattice, analogous decompositions to Figure 4 can be made (see [7]) and algebraic equations can be deduced. Another simple argument is to go from a strict heap of dimers to a general heap by substituting to a single dimer a *pile* of dimers as shown on Figure 5.

When applied to pyramid of half-pyramid, this substitution implies the following relation between the corresponding generating functions:

$$P_T(t) = P_S\left(\frac{t}{1-t}\right); Q_T(t) = Q_S\left(\frac{t}{1-t}\right). \tag{1.5}$$

From equations (1.4) and (1.5) one deduces the formulae (1.2).

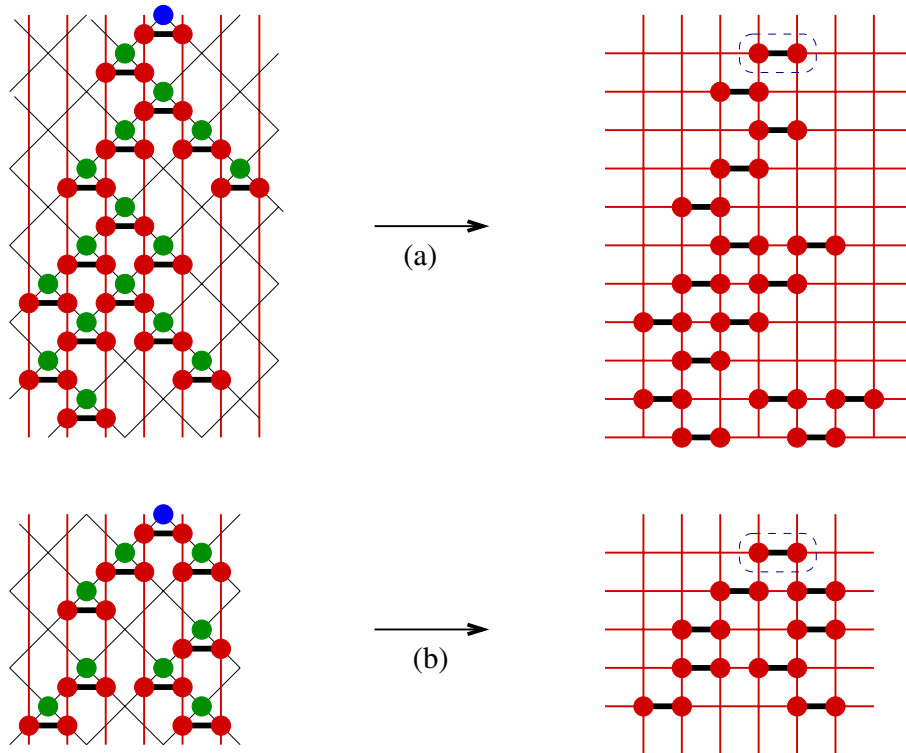


Figure 3. Bijection between directed animals and pyramids of dimers (a) square lattice and strict pyramids (b) triangular lattice.

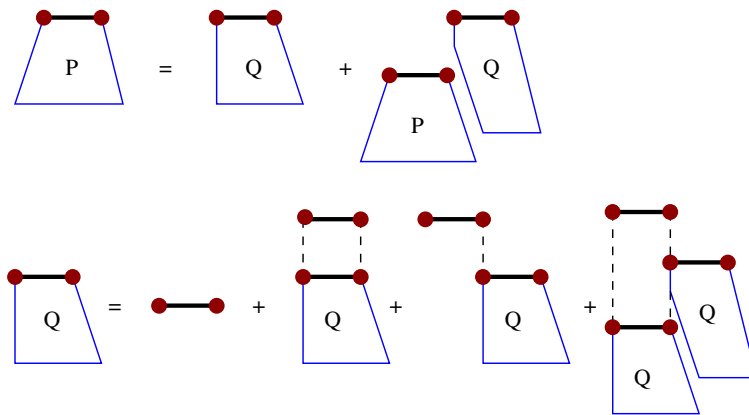


Figure 4. Factorisation of pyramid and half-pyramids.

2. Multi-directed animals

A heap of dimers is called *connected* when no empty column occurs between the leftmost and rightmost non empty columns of the heap. In particular the orthogonal projection of a heap of dimers is a connected segment of \mathbb{Z} . An example is displayed on Figure 6. Any heap of dimers can be decomposed into a sequence of connected heaps, called its *connected components*, separated by empty columns. See Figure 7.

The transformation V described in section 1 can be done for every *animal* on a square or triangular lattice, as shown on Figure 6. Recall that an *animal* is a *connected* set of points of the square (or

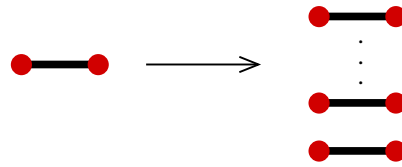


Figure 5. From strict heaps to general heaps.

triangular) lattice, that is, such that every pair of points (or cell) can be connected by a path formed by points in the animal and with elementary steps going to a next neighbour point. The transformation V is not bijective. The result $V(A)$, for any animal is always a connected heap.

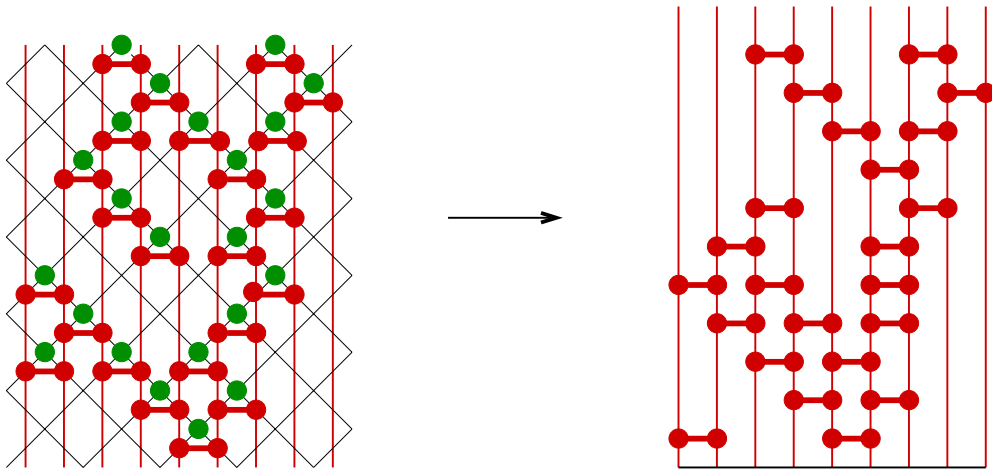


Figure 6. From animals to connected heaps of dimers.

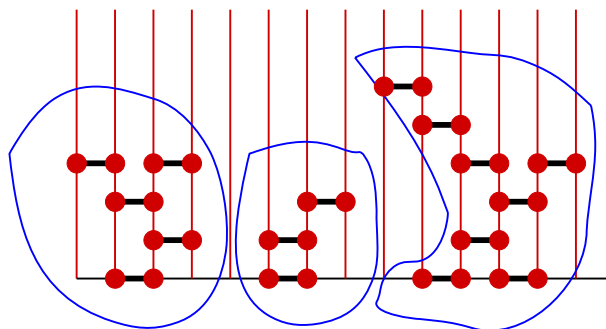


Figure 7. Connected components of a heap of dimers.

Bousquet-Mélou and Reznitzer have introduced in [7] the class of *multi-directed animals* so that the map V becomes a bijection between connected heap of dimers and that class of animals. For H a connected heap, let m_1, \dots, m_k be the maximal pieces of H , ordered from left to right. We successively apply the push operator to the pieces m_1, \dots, m_k , giving rise to a sequence of pyramids P_1, \dots, P_k . Each pyramids P_i is in bijection via the map V with a directed animal A_i . Then we glue successively the

directed animals A_1, \dots, A_k , by putting each animal A_i “far above” the previous animals A_1, \dots, A_{i-1} already glued, and sliding it down until it connects with the previous glued animals. This process is shown on Figure 8. The resulting set of points is called a *multi-directed animal*.

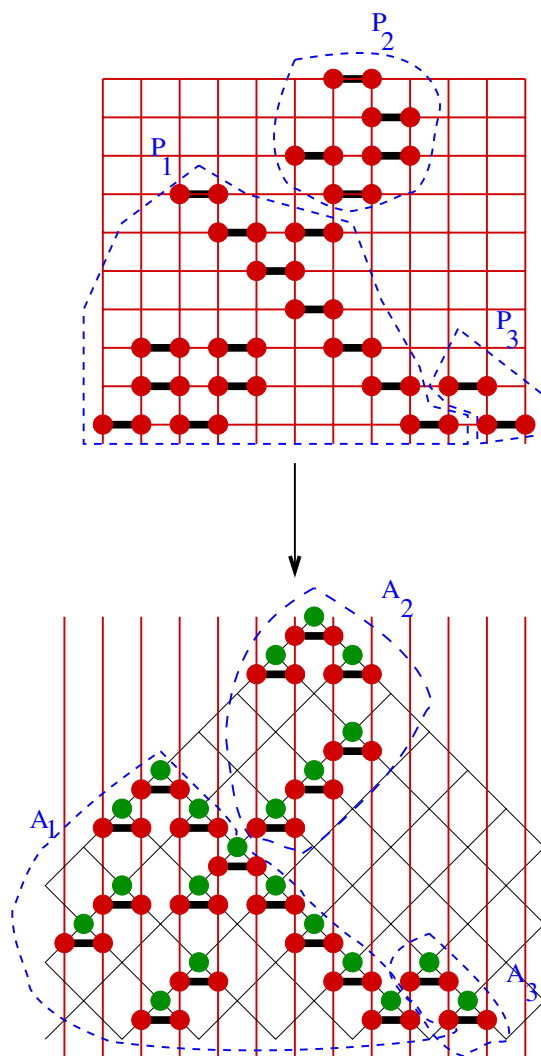


Figure 8. From connected heaps to multi-directed animals.

It would also be possible to defined multi-directed animals on a square lattice, in bijection with connected strict heaps of dimers, see [7].

One of the main result of [7] is an explicit expression for the generating function of multi-directed animals (enumerated by number of cells), or equivalently of connected heaps of dimers (enumerated by number of dimers, and up to an horizontal translation).

Proposition 1. *Let Q denote the generating function for general (resp. strict) half-pyramids. Then the generating function C for general (resp. strict) connected heaps of dimers is given by the following relation*

$$C = \frac{Q}{(1 - Q) \left(1 - \sum_{k \geq 1} \frac{Q^{k+1}}{1 - Q^k(1+Q)}\right)}. \tag{2.1}$$

As shown in [7], this series is not algebraic, nor even D-finite. Recall that a power series is *D-finite* iff its coefficients a_n satisfies a linear recurrence relation with polynomial coefficients in n . The purpose of this paper is to give a bijective proof of identity (2.1).

3. Preliminaries about heaps

We have an inversion lemma, giving the generating function of heaps of pieces in general, see [8] and [21]. In the case of heaps of dimers on a segment of length n , this lemma can be written in the following way.

Let $U_n(t)$ be the n^{th} *Tchebychef polynomial* of the second kind, that is the polynomial defined by the relation $\sin(n + 1)\theta = (\sin\theta)U_n(\cos\theta)$.

We denote by $F_n(t^2)$ the reciprocal of the polynomial $U_n(t/2)$. These polynomials (sometimes called *Fibonacci polynomials*) can also be defined by the classical three terms recurrence relation:

$$F_{n+1}(t) = F_n(t) - tF_{n-1}(t) \text{ for } n \geq 1 \text{ with } F_0(t) = F_1(t) = 1. \tag{3.1}$$

These Fibonacci polynomials can be interpreted combinatorially as the (alternating in sign) generating polynomial for *trivial heaps* of dimers on a segment, that is heaps where all the pieces are at level 0. These are classical combinatorial objects also known as *matchings*. The Fibonacci polynomial is also the reciprocal (up to change of t^2 into t) of the so-called *matching polynomial* of the segment graph. An example is displayed on Figure 9.

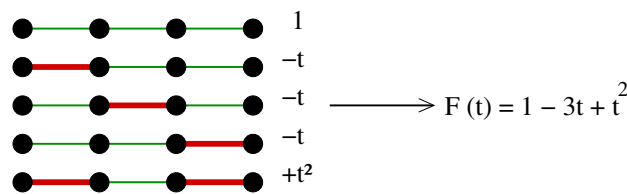


Figure 9. The Fibonacci polynomial $F_4(t)$ and trivial heaps of dimers on a segment.

Proposition 2. Inversion lemma. *The generating function for heaps of dimers (counted according to the number of dimers) on a segment of n points (i.e. heaps of dimers in a bounded strip of width $n - 1$) is the inverse of the alternating generating function for trivial heaps of dimers on this segment, i.e. $1/F_n(t)$.*

In this paper we will also need to introduce the notion of *left width* of a pyramid and its related generating function.

Let P be a pyramid of dimers. The number of non empty columns at the left of the column containing the maximal piece of P is called the *left width* of the pyramid. Let P_k be the generating function of pyramids with left width $k \geq 0$. The power series is given by the following simple formula

$$P_k = Q^{k+1}. \tag{3.2}$$

Figure 10 displays the idea for a bijective proof of this formula. If $k = 0$, then the pyramid P is reduced to a single half-pyramid. For $k \geq 1$, let m_1, \dots, m_k be the maximal pieces in the k columns located at the left of the column containing the maximal piece of P (numbered from left to right). We apply successively the push operator to m_1, \dots, m_k , obtaining a sequence of half-pyramids. At the end remains a single half-pyramid. The pyramid P has been decomposed into a product of $(k + 1)$ half-pyramids and formula (3.2) is proven.

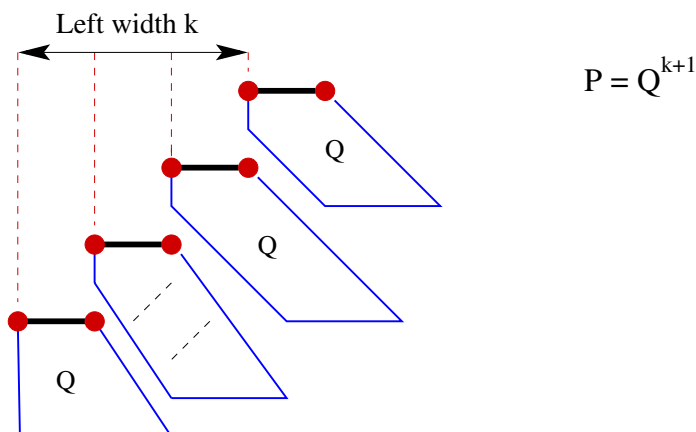


Figure 10. Decomposition of pyramids into half-pyramids.

4. The Nordic decomposition of a connected heap of dimers

The “*Nordic decomposition*” is a recursive decomposition of a connected heap of dimers. The name “Nordic” comes from the fact that we found this decomposition during our stay in Iceland and Sweden, beginning at the 24th Nordic (and first Franco-Nordic) meeting in Reykjavik and then during our stay at the Mittag-Leffler Institute in January and February 2005.

Let H be a non-empty connected heap of dimers. Let α be the rightmost maximal dimer. The push operator factorizes the heap H into a product $P \otimes H'$, where H' is a heap and P is a pyramid having α as maximal piece. The heap H' could be empty. In that case H is reduced to the pyramid P . If H' is not empty, we continue in the following way.

The heap H' is not necessarily connected, but can be factorized into its connected components H'_1, \dots, H'_r (written from left to right). Let $G = H'_1$ be its leftmost connected component, and let J be the heap formed by the remaining connected components H'_2, \dots, H'_r . Between the rightmost non-empty column of G and the column containing the dimer α , there exist at least $k \geq 1$ columns, numbered from left to right C_1, \dots, C_k . If this was not the case, we would have $r = 1$, $G = H'$ and there would be some dimers of G inside the column just at the left of the column containing the dimer α , and above this dimer α . This would contradict the fact that α is a maximal piece of H .

It may be possible that $C_1 = C_k$ (this is the case $k = 1$). The heap J (may be empty) is located between the columns C_1 and C_k . It is an arbitrary heap of dimers inside a strip of width $k - 2$.

Inside the column C_1 , there are some dimers of the pyramid P (if this was not the case the heap $H = P \otimes G \otimes J$ would not be connected). Let β be the topmost dimer of the pyramid P inside the column C_1 . Again, applying the push operator on the dimer β , we can decompose the pyramid P into a product of two pyramids $P = P' \otimes P''$. The pyramid P' has β as maximal piece. The pyramid P'' has α as maximal piece and has a leftwidth equal to $k - 1$.

To resume the situation, if the heap H is not reduced to a single pyramid P , we have splitted the connected heap H into four disjoint heaps: a connected heap G , a heap J on a segment of length $k - 2$ (empty if $k = 1$ or 2), a pyramid P' and another pyramid P'' of leftwidth $k - 1$. Each of these heaps is defined up to a translation (parallel to the horizontal axis).

Conversely, from the knowledge of the quartet G, J, P', P'' where the heaps are defined up to an horizontal translation, we can reconstruct the heap H in the following way: first the pyramid P is obtained by taking the product of the pyramid P' with a pyramid P'' translated such that the maximal piece β of the pyramid P' is in a column just at the left of the leftmost non-empty column of P'' . Then, we put above the heap G , translated such that its rightmost non-empty column is located just at the left of the column containing the maximal piece β of the pyramid P' . Finally we put above all this the heap

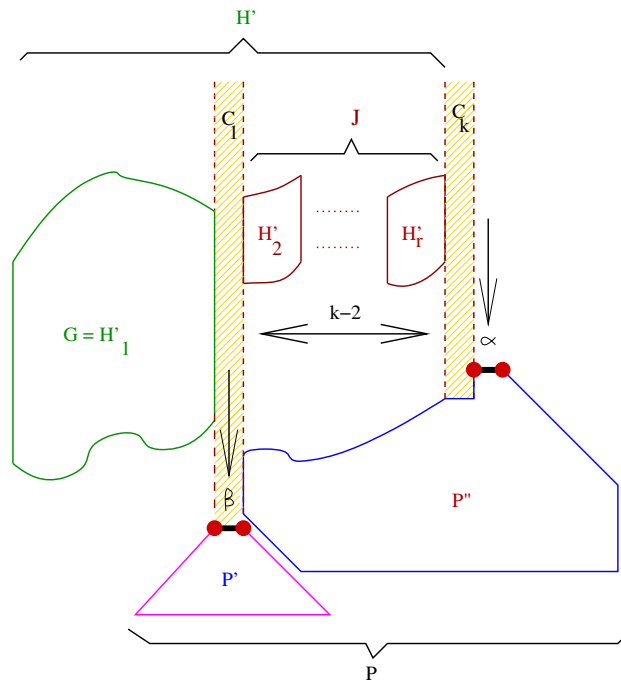


Figure 11. The Nordic decomposition of a connected heap of dimers.

J , translated such the segment of length $k - 2$ on which J is defined, is the segment corresponding to the points located at distance $1, 2, \dots, k - 2$ and at the left of the maximal piece α of the pyramid P'' .

The Nordic decomposition of a connected heap into four pieces leads to the following identity for the generating functions

$$C = \frac{Q}{1 - Q} + C \sum_{k \geq 1} \frac{Q}{1 - Q} \times Q^k \times \frac{1}{F_{k-1}}. \tag{4.1}$$

The generating function for the pyramid P' is $Q/(1 - Q)$, the generating function of the pyramid P'' with left width $k - 1$ is Q^k , while the generating function for the part J is $1/F_{k-1}(t)$.

From (4.1) we deduce the following expression for the generating function for connected heaps of dimers

$$C = \frac{Q}{(1 - Q) \left(1 - \sum_{k \geq 1} \frac{Q^{k+1}}{(1 - Q)F_{k-1}} \right)}. \tag{4.2}$$

This is not exactly the formula given by Bousquet-Mélou and Rechnitzer, but the Fibonacci polynomials can be expressed in term of the generating function $Q(t)$ for Catalan numbers (or half-pyramids) in the following way

$$F_n(t) = \frac{(1 - Q^{n+1})}{(1 - Q)(1 + Q)^n}. \tag{4.3}$$

Replacing $F_{k-1}(t)$ in identity (4.2) gives the following identity

$$C = \frac{Q}{(1 - Q) \left(1 - \sum_{k \geq 1} \frac{Q^{k+1}(1+Q)^{k-1}}{(1-Q^k)}\right)}. \tag{4.4}$$

which is equivalent to identity (2.1), as shown in [7].

The same Nordic decomposition would applies for strict connected heaps of dimers. The corresponding identities should be replaced with generating functions for strict heaps. There would be an analogue for the generating function $1/F(t)$, with an analogue for identity (4.3), which can be proved bijectively using the same proof as below in section 5.

5. Fibonacci polynomials and Catalan numbers

For completeness, we give a bijective proof of the relation (4.3) relating Fibonacci polynomials and Catalan numbers. First we write the identity (4.3) in the following form

$$(1 + Q)^n = \frac{1}{F_n(t)} \times (1 + Q + \dots + Q^n), \text{ for every } n \geq 1. \tag{5.1}$$

We claim that both sides of identity (5.1) are the generating function for heaps of dimers H on the non negative integers subject to the condition that the maximal pieces of H are located in the first n columns over the horizontal axis. In other words, the projection of these maximal dimers are contained in the segment $[0, n]$.

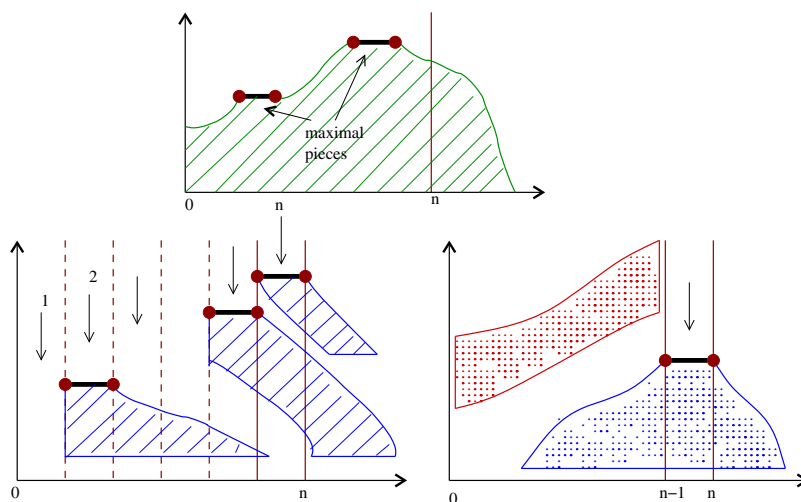


Figure 12. Bijective proof of (5.1).

The left handside of identity (5.1) is obtained by reading from left to right the first n columns over the non negative integers. Some columns may not contain any dimers of H , but we can always apply the push operator for each non-empty column. We thus decompose the heap H into a product of n (possibly empty) half-pyramids. This gives the interpretation of the left handside of (5.1).

In order to interpret the right handside, we apply the push operator on the maximal piece of the column $(n - 1, n)$ (if the column is non empty). We get a (possibly empty) pyramid having left width $\leq n$. The heap H is thus factorized into this pyramid and a heap on the segment $[0, n - 1]$. Applying the inversion lemma, we get the left handside of identity (5.1).

As mentioned at the end of the previous section, analogue proof would applies for strict heaps, and thus we obtain bijectively identity (2.1) for square lattice multi-directed animals, i.e. connected heaps of dimers.

6. Relation with Lorentzian triangulations

One possible interest of this bijective proof is in relation with 2D Lorentzian quantum gravity.

Lorentzian triangulations have been introduced by Ambjørn and Loll in [1] as a model for 2D quantum gravity, and further study by Di Francesco, Guitter and Kristjansen [11]. A (*general*) *Lorentzian triangulation* is a triangulation on the plane, where all the vertices of the triangles relies on parallel horizontal lines (corresponding to time $T = 0, 1, \dots$). Between two lines, we have a connected sequence of triangles, pointing upwards or downwards. See Figure 13 (a). Physicists have considered some border conditions. In [1], the first triangle in each row is pointing upwards. In the paper [11], the triangulations satisfy two border conditions: the first (resp. last) triangle in each row is pointing upwards (resp. downwards). See Figure 13 (b) and (c).

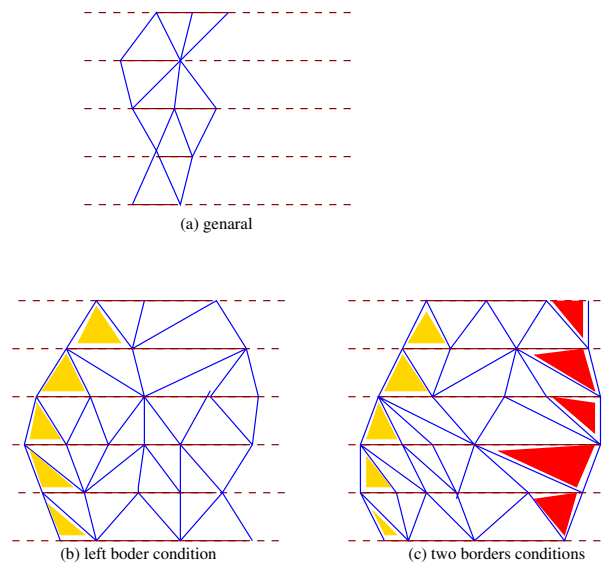


Figure 13. Lorentzian triangulations.

Computing some physical quantities, such as the so-called “*loop propagator from one geometry to another*” are equivalent to the combinatorial problems of enumerating these Lorentzian triangulations according to various parameters such as the number of triangles, the number of rows, the number of points on the lower and upper constant-time lines. In [11], an extra parameter “*curvature*” is added. Explicit expressions are given in [1] for such generating functions with the first border condition and in [11] with the two border conditions and involving the curvature parameter. The general case did not seem to have been studied in the physics literature. In fact, general 2D Lorentzian triangulations are nothing but connected heaps of dimers ...

A bijection can be given (see [11]). Let H be a connected heap of dimers. Replace each dimer by a pair of triangles as shown in Figure 14. Then glue together in each column these triangles. Deleting the triangles of the first and last column gives rise to a Lorentzian triangulation (rotated by 90°). In fact it is not an arbitrary Lorentzian triangulation: it is a triangulation with no *articulation point*, that is each constant-time line contains at least two vertices of some triangle.

Conversely, start from such a triangulation with no articulation point. Rotate 90° . Add an extra sequence of triangles on both sides so that the first and last constant-time line contains only one vertex. Take the *dual map* (in the language of *planar graphs*). From this dual map can be extracted edges leading to a heap of dimers after some deformation, as shown on Figure 14.

In [1], the border condition is equivalent to say that the heap is a half-pyramid. The corresponding generating function is thus related to the Catalan generating function. Complete bijective proofs can be constructed for the formulae involved in [1] and [11], see James [14] and James, Viennot [15].

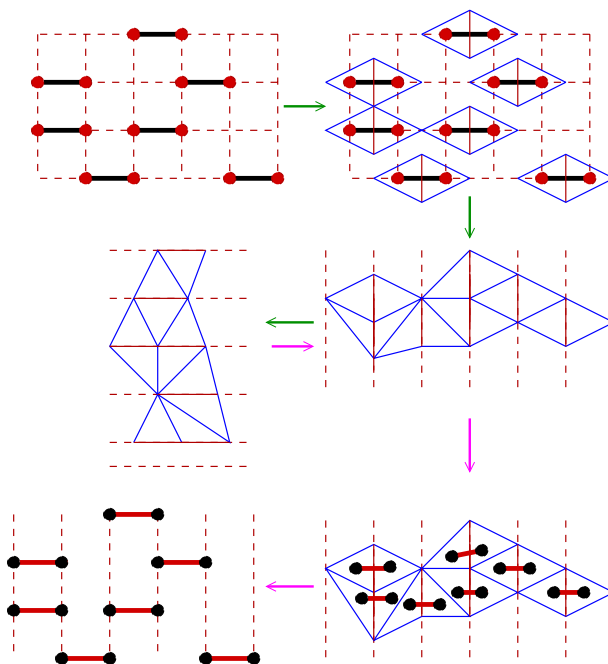


Figure 14. Bijection between connected heaps of dimers and Lorentzian triangulations (with no articulation point).

Note that a general Lorentzian triangulation can be uniquely decomposed as a sequence of Lorentzian triangulations without articulations points. Thus the formula of Bousquet-Mélou and Rechnitzer solves the problem of the enumeration of general Lorentzian triangulations counted by the number of triangles (in fact up to the slight modification that some extra triangles have been added to both sides of the configuration). The formula shows that 2D Lorentzian gravity in the general case is not in the same universality class than the model considered in [1] and [11] with the above border conditions.

A formula for the four variables generating function is in fact given in Bousquet-Mélou and Rechnitzer [7] (up to some changes of variables), different bijections are involved. Nevertheless, some open problems remain for these general 2D Lorentzian triangulations, such as the enumeration problem with the curvature parameter. It is our hope that the Nordic decomposition of a connected heap of dimers presented here will be useful for approaching these open problems.

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References

- [1] Ambjørn J and Loll R 1998 Non-perturbative Lorentzian quantum gravity, causality and topology change *Nucl. Phys. B* **536** 407–434.

- [2] Barcucci E, Del Lungo A, Pergola E and Pinzani R 1999 Directed animals, forests and permutations *Discrete Math.* **204** 41–71
- [3] Bétréma J and Penaud J-G 1993 Animaux et arbres guingois *Theor. Comp. Sci.* **117** 67–89
- [4] Bétréma J and Penaud J-G 1993 Modèles avec particules dures, animaux dirigés et séries en variables partiellement commutatives *Preprint* arXiv:math.CO/0106210 (Technical Report 93-18, LaBRI, Université Bordeaux 1)
- [5] Bousquet-Mélou M 1996 Combinatoire énumérative *Mémoire d'Habilitation* (Technical Report 1154-96, LaBRI, Université Bordeaux 1)
- [6] Bousquet-Mélou M and Viennot X G 1992 Empilements de segments et q-énumération de polyominoes convexes dirigés *J. Combinatorial Theory Ser. A* **60** 196–224
- [7] Bousquet-Mélou M and Rechnitzer A 2002 Lattice animals and heaps of dimers *Discrete Maths.* **258** 235–274
- [8] Cartier P and Foata D 1969 Problèmes combinatoires de commutation et réarrangements *Lecture Notes in Math.* **85** 88 pages (Berlin: Springer-Verlag)
- [9] Dhar D 1982 Equivalence of the two-dimensional directed-site animal problem to Baxter's hard square lattice gas model *Phys. Rev. Lett.* **49** 959–962
- [10] Dhar D 1993 Exact solution of a directed-site animal-enumeration problem in three dimensions *Phys. Rev. Lett.* **51** 853–856
- [11] Di Francesco P, Guitter E and Kristjansen C 2000 Integrable 2D Lorentzian gravity and random walks *Nucl. Phys. B* **567** 515–553
- [12] Gouyou-Beauchamps D and Viennot G 1988 Equivalence of the two-dimensional directed animal problem to a one dimensional path problem *Adv. in Appl. Math.* **9** 334–357
- [13] Hakim V and Nadal J-P 1983 Exact results for 2D directed animals on a strip of finite width *J. Phys. A: Math. Gen.* **16** L213–L218
- [14] James W 2005 Extending exact solutions to discrete lattice models (Thesis submitted, University of Melbourne)
- [15] James W and Viennot X G Heaps of segments and 2D Lorentzian quantum gravity *in preparation*
- [16] Jensen I and Guttmann A J Statistics of lattice animals (polyominoes) and polygons *J. Phys. A: Math. Gen.* **33** L257–L263
- [17] Nadal J-P, Derrida B and Vannimenus J 1982 Directed lattice animals in 2 dimensions: numerical and exact results *J. Physique* **43** 1561–1574
- [18] Penaud J-G Arbres et animaux *Mémoire d'habilitation* (Université Bordeaux 1)
- [19] L. W. Shapiro, *Directed animals and Motzkin paths*, preprint, Howard University, Washington DC, 1999.
- [20] G. Viennot, *Problèmes combinatoires posés par la physique statistique*, Séminaire Bourbaki n° 626, Février 1984, in *Astérisque* 121-122 (1985), Soc. Math. France.
- [21] G. X. Viennot, *Heaps of pieces 1: basic definitions and combinatorial lemma*, in *Combinatoire énumérative*, eds G. Labelle and P. Leroux, Lecture Notes in Maths, 1234 (1986) 321-246.